Distributed Model Predictive Control for Optimal Output Consensus of Multi-agent Systems over Directed Graphs ⋆

Nan Bai [a], Qishao Wang [b], Zhisheng Duan [a], Changxin Liu [c]

[a] *Department of Mechanics and Engineering Science, Peking University, Beijing, China*

[b] *Department of Dynamics and Control, Beihang University, Beijing, China*

[c] *School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, Stockholm, Sweden*

Abstract

In this paper, a distributed model predictive control (MPC) scheme is established to solve the optimal output consensus problem of heterogeneous multi-agent systems over directed graphs. Within the framework of MPC, we take both the control input and the consistent output state as decision variables to formulate a constrained optimization problem. Inspired by the primal decomposition technique and the push-sum dual average method, a distributed algorithm is designed to address the optimization problem. The convergence analysis of the proposed algorithm is given, which shows the convergence properties related to the number of iterations. Then, considering the limited computational resources in practical applications, an improved MPC-based approach with premature termination is further developed. The closed-loop stability is analyzed under the suboptimal MPC framework, deriving appropriate terminal conditions to guarantee the asymptotic consensus of multi-agent systems. Finally, numerical simulations demonstrate the effectiveness of the theoretical results.

Key words: Model predictive control; multi-agent system; consensus; distributed optimization.

1 Introduction

The collaboration of the multi-agent system has attracted considerable attention in recent years due to its potential for increasing efficiency. There have been substantial efforts focusing on this issue, boosting the development of its applications such as smart grids [1], sensor fusion [2], multi-vehicle coordination [3], and resource allocation [4]. One of the fundamental and significant topics is consensus, which aims to design appropriate control protocols for agents to reach agreement of common interests. As a pioneering work, the theoretical framework of the consensus-seeking problem for multi-agent systems is formulated by [5], where the individual dynamics are first-order integrators. Subsequently, more complex dynamical models have been investigated [6,7,8,9] to accommodate diverse real-world systems. In addition, some work has been extended to solve the output consensus problem [10,11,12,13] to fulfill variety of requirements.

It is noteworthy that the aforementioned investigations mainly focus on unconstrained systems. However, constraints are ubiquitous in practical applications due to the restriction on machine's capability and safety or some other factors. Model predictive control (MPC) is a well-established framework to deal with constraints systematically in control problems by solving a constrained optimization problem at each time to obtain applicable control inputs. For multi-agent systems, distributed MPC (DMPC) can explicitly promote collaboration among the agents. In [14], a DMPC approach is proposed for the multi-agent system containing coupled dynamics and independent constraints, where agents' common final consensus state is set as the origin. In [15], the collaborative task is extended from consensus to formation for dynamically decoupled agents, where the collision avoidance constraints between agents are also taken into account. Additionally, the global coupled constraints of system states and control inputs are efficiently handled by [16]. To save communication resources, a distributed event-triggered MPC scheme is developed by [17], which effectively reduces communica-

⋆ This paper was not presented at any IFAC meeting. Corresponding author: Z. Duan.

Email addresses: bainan@stu.pku.edu.cn (Nan Bai), wangqishao@buaa.edu.cn (Qishao Wang), duanzs@pku.edu.cn (Zhisheng Duan), changxin@kth.se (Changxin Liu).

tion to facilitate practical applications.

Nevertheless, existing studies neglect the consensus point, or only concentrate on the predefined final state, thereby compromising the control performance to a certain extent. To complete the consensus task while improving the system performance, i.e., achieving the optimal consensus state of multi-agent systems, an intuitive idea is to treat both the final consensus state and the control input as decision variables to be optimized. Following this intuition, a DMPC framework for simultaneous optimization of final consensus state and linear quadratic performance is proposed in [18], where the optimization problem can be effectively solved by [19]. However, this approach cannot deal with constraints and fails to guarantee rigorous closed-loop stability for heterogeneous agents. Another attempt is employed by [20] to address these difficulties, where the performance is also described as the linear quadratic index and the incremental subgradient method in [21] is applied. It should be pointed out that all of the above methods have stringent requirements in terms of communication topology. Specifically, [18,19] can only be utilized over undirected graphs, while the methods in [20,21] require the communication topology to be either a cycle or a complete graph, all of which strictly restrict practical implementation. To the best of the authors' knowledge, it still remains an open problem to solve the optimal output consensus problem over general directed graphs.

Inspired by the aforementioned progress, this paper deals with the optimal output consensus problem of heterogeneous multi-agent systems with constraints over directed graphs. The contributions of this paper can be summarized as follows. Firstly, a DMPC framework applicable to directed graphs is established, where both the control input and the consistent output state are considered as decision variables to formulate a constrained optimization problem at each time. In this way, it is possible to optimize the system performance and the consensus output simultaneously. Secondly, the primal decomposition technique and push-sum dual average are utilized to design the distributed iterative algorithm to solve the optimization problem under directed graphs. Finally, an improved MPC-based scheme is further developed with premature termination to save computational resources. The corresponding terminal conditions have also been derived to guarantee the closed-loop stability and ensure the asymptotic consensus of multi-agent systems.

Notation: $\mathbb{R}^n$ denotes the set of n-dimensional real column vectors; $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ real matrices; $\mathrm{col}\{x_1, x_2, \cdots, x_n\} \triangleq \left[x_1^\top, x_2^\top, \cdots, x_n^\top\right]^\top$ denotes the collection of vectors $x_i \in \mathbb{R}^{n_i}, i \in \{1, 2, \cdots, n\}$; $\mathbf{1}$ and $\mathbf{0}$ denote column vectors with all elements of 1 and 0 in proper dimensions, respectively; for a matrix $\mathcal{A}$, $\mathcal{A}_{ij}$ denotes the element located in the ith row and jth column of $\mathcal{A}$. $\mathrm{rand}(a, b)$ represents a random number with a uniform distribution on the interval $[a, b]$; for constant a, $\lfloor a \rfloor$ denotes the largest integer that is less than or equal to a; $\otimes$ denotes the Kronecker product of matrices; $\|\cdots\|$ denotes the Euclidean norm of vectors; $\langle \cdots, \cdots \rangle$ denotes the inner product of vectors; for matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times m}$, $\|A\|_B^2 \triangleq A^\top BA$.

2 Problem Formulation and Preliminaries

2.1 Problem Statement

Considering a network system containing N heterogeneous agents, the dynamics of agent i can be described as

$$\begin{aligned} x_i(k+1) &= A_i x_i(k) + B_i u_i(k), \\ y_i(k) &= C_i x_i(k), \\ x_i(k) &\in \mathcal{X}_i,\ u_i(k) \in \mathcal{U}_i, \end{aligned} \tag{1}$$

where $A_i \in \mathbb{R}^{n_i\times n_i}$, $B_i \in \mathbb{R}^{n_i\times m_i}$, $C_i \in \mathbb{R}^{r\times n_i}$, $\mathcal{X}_i$ is a convex and closed subset of $\mathbb{R}^{n_i}$, and $\mathcal{U}_i$ is a convex and compact subset of $\mathbb{R}^{m_i}$.

Assumption 1 *For each agent, (A_i, B_i) is controllable.*

The communication between agents can be described by a directed graph $\mathcal{G}$ containing a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$. A directed edge $(i, j) \in \mathcal{E}$ denotes that agent i can transmit information to agent j directly. The adjacency matrix $\mathcal{A} \in \mathbb{R}^{N\times N}$ is defined as $\mathcal{A}_{ij} = 1$ if and only if $(j, i) \in \mathcal{E}$ and $\mathcal{A}_{ii} = 1$ holds by nature. The in-neighbor and the out-neighbor of agent i are defined as $\mathcal{N}_i^{\mathrm{in}} = \{j \mid (j, i) \in \mathcal{E}\}$ and $\mathcal{N}_i^{\mathrm{out}} = \{j \mid (i, j) \in \mathcal{E}\}$, respectively. Define ν_i and μ_i to represent the out-degree and in-degree of agent i, that is, $\nu_i = \sum_{j=1}^{N} \mathcal{A}_{ij}$ is the cardinality of $\mathcal{N}_i^{\mathrm{out}}$ and $\mu_i = \sum_{j=1}^{N} \mathcal{A}_{ji}$ is the cardinality of $\mathcal{N}_i^{\mathrm{in}}$, respectively.

Assumption 2 *The directed communication topology graph $\mathcal{G}$ is strongly connected.*

This paper focuses on the optimal output consensus problem of the heterogeneous multi-agent system while ensuring that each individual converges to a stable state. Specifically, the control objective can be described as $\lim_{k\to\infty}(y_i(k) - y_j(k)) = \mathbf{0}$, $\forall i, j \in \{1, 2, \ldots, N\}$, and $\lim_{k\to\infty}(x_i(k), u_i(k)) = (z_i^e, u_i^e)$, $\forall i \in \{1, 2, \ldots, N\}$. Here, (z_i^e, u_i^e) represents a steady solution for each agent, and (z_i^e, u_i^e, y_i) satisfies the equation

$$(A_i - I_{n_i}) z_i^e + B_i u_i^e = \mathbf{0}, \tag{2a}$$
$$C_i z_i^e - y_i = \mathbf{0}. \tag{2b}$$

It can be followed from the PBH test for controllability [22, Theorem 12.3] that (2) has non-trivial solutions due to the controllability in Assumption 1.

Then, the objective of this paper can be transformed to design proper distributed control method such that

$$\lim_{k\to\infty} x_i(k) = z_i^e,\ \lim_{t\to\infty} u_i(t) = u_i^e,\ \forall i \in \{1,2,\ldots,N\},$$
$$\lim_{k\to\infty} y_i(k) = \lim_{t\to\infty} y_j(k) = y, \quad \forall i,j \in \{1,2,\ldots,N\},$$

where (z_i^e, u_i^e, y) satisfies (2). Subsequently, define the set of steady point and the feasible consensus output of the multi-agent system as follows.

Definition 1 *The set of steady point of each agent is defined as $\mathscr{E}_i = \{(u_i^e, z_i^e) | (z_i^e, u_i^e)$ satisfying (2a)$\}$.*

Definition 2 *The set of feasible consensus outputs is defined as $\mathscr{Y} = \{y \,|\, z_i^e \in \mathcal{X}_i, u_i^e \in \mathcal{U}_i, (z_i^e, u_i^e, y)$ satisfying (2), $\forall i \in \{1,2,\ldots,N\}.\}$.*

Assumption 3 *For each agent, the intersection of $\mathcal{U}_i \times \mathcal{X}_i$ and $\mathscr{E}_i$ is non-empty. Moreover, the set of feasible consensus outputs $\mathscr{Y}$ is non-empty.*

2.2 MPC for Optimal Output Consensus

In what follows, the MPC framework will be utilized to enable agents to plan and reach the optimal steady consistent output by negotiation. Assume that t_k is the real time of the kth prediction, $\mathcal{T}$ is the finite prediction horizon, and $\Delta t = t_{k+1} - t_k$ is the control period satisfying $0 < \Delta t \leq \mathcal{T}$. Then, at any time t_k, recalling the constraint on the steady state in (2), an additional constraint on the last predicted state will be involved to force the system to reach a steady state in $\mathscr{E}_i$, i.e.,

$$x_i(\mathcal{T}|t_k) = A_i x_i(\mathcal{T}|t_k) + B_i u_i(\mathcal{T}|t_k),$$
$$y(\mathcal{T}|t_k) = C_i x_i(\mathcal{T}|t_k),$$

where $u_i(\mathcal{T}|t_k)$ and $x_i(\mathcal{T}|t_k)$ represent the predicted input and system state of agent i at time instant $\mathcal{T} + t_k$. Correspondingly, define the cost function of each individual as

$$J_i(U_i(t_k)) = \sum_{t=0}^{\mathcal{T}-1} \Big[\|x_i(t|t_k) - x_i(\mathcal{T}|t_k)\|_{Q_i}^2 + \|u_i(t|t_k) - u_i(\mathcal{T}|t_k)\|_{R_i}^2 \Big],$$

where Q_i and R_i are positive definite weighting matrices, and $U_i(t_k) = \mathrm{col}\{u_i(0|t_k), u_i(1|t_k), \ldots, u_i(\mathcal{T}|t_k)\}$. On this basis, at each time t_k, consider the following optimization problem within the prediction horizon $\mathcal{T}$.

Problem 1 *At each time t_k, with the given initial state $x_i(0|t_k) = x_i(t_k) \in \mathcal{X}_i$, the global optimization problem is formulated as*

$$\min_{\mathbf{U}(t_k), y(t_k)} J(\mathbf{U}(t_k)) = \sum_{i=1}^{N} J_i(U_i(t_k))$$

$$\begin{aligned}
\text{s.t.}\quad x_i(k+1|t_k) &= A_i x_i(k|t_k) + B_i u_i(k|t_k), && \text{(3a)}\\
y_i(k|t_k) &= C_i x_i(k|t_k), && \text{(3b)}\\
x_i(k|t_k) &\in \mathcal{X}_i, k \in \{0,1,\ldots,\mathcal{T}\}, && \text{(3c)}\\
u_i(k|t_k) &\in \mathcal{U}_i, k \in \{0,1,\ldots,\mathcal{T}-1\}, && \text{(3d)}\\
x_i(\mathcal{T}|t_k) &= A_i x_i(\mathcal{T}|t_k) + B_i u_i(\mathcal{T}|t_k), && \text{(3e)}\\
y_i(\mathcal{T}|t_k) &= C_i x_i(\mathcal{T}|t_k) = y(t_k) \in \mathscr{Y}, && \text{(3f)}\\
x_i(0|t_k) &= x_i(t_k),\ i \in \{1,2,\ldots,N\}, && \text{(3g)}
\end{aligned}$$

where $\mathbf{U}(t_k) = \mathrm{col}\{U_1(t_k), U_2(t_k), \ldots, U_N(t_k)\}$.

Remark 1 *The equations (3e)-(3f) are additional terminal constraints to restrict the final output $y_i(\mathcal{T}|t_k)$ to be consistent in the prediction horizon $\mathcal{T}$, which play an important role to design the algorithm and guarantee the stability of the closed-loop system.*

Assumption 4 *The prediction horizon $\mathcal{T}$ is large enough such that all the feasible consensus output in $\mathscr{Y}$ can be achieved by every agent. In other words, for all $y \in \mathscr{Y}$, there exists a feasible control input sequence $\mathrm{col}\{u_i(0), u_i(1), \ldots, u_i(\mathcal{T}-1)\}$ in the relative interior of $\mathcal{U}_i$ such that $y_i(\mathcal{T}) = y$ for each individual.*

3 Distributed Optimization-Based Algorithm

Solving the optimization problem (3) in a distributed manner is a challenging task for two reasons. First, both the control input U_i and the consistent output y are decision variables, while y is coupled in the consensus constraint (3f) between agents. Second, agents are only allowed to communicate through a directed graph, which cannot be handled by existing methods such as [18,19,20,21]. In this section, we overcome these difficulties to design a fully distributed algorithm.

3.1 Primal Decomposition and Algorithm Design

Before designing the algorithm to solve (3) in the MPC scheme, the primal decomposition is first performed for Problem 1 to explore its useful properties.

For any $y(t_k) \in \mathscr{Y}$, define a new function for each agent

$$f_i(y(t_k)) = \min_{U_i(t_k)} J_i(U_i(t_k)),\ \text{s.t.}(3a) - (3g),$$

where $y(t_k)$ and $U_i(t_k)$ can be viewed as independent variables in the cost function owing to (3e)-(3f). Then, the optimization problem (3) can be rewritten as

$$\begin{aligned}
&\min_{y(t_k)} F(y(t_k)) = \sum_{i=1}^{N} f_i(y_i(t_k)) \\
&\quad \text{s.t. } y_i(t_k) = y(t_k) \in \mathscr{Y}.
\end{aligned} \tag{4}$$

After solving (4) to obtain the optimal consensus output $y^\star(t_k)$, the corresponding optimal control input sequence $U_i^\star(t_k)$ of each agent can be easily calculated by solving the constrained quadratic optimization problem (3) with $y(t_k) = y^\star(t_k)$ using existing methods such as quadratic programming [23,24,25]. In what follows, the distributed approach to solve (4) will be investigated. Considering the problem formulation, there have been some distributed (sub)gradient-based methods [26,27,28,29]. However, none of these results clarify how to obtain specific information about the (sub)gradient of the objective function. For the sake of designing the subgradient-based algorithm to solve the practical problem (4), the following observations specify its subgradient information and more properties.

Proposition 1 *The cost function $f_i(y)$ defined in (4) is a convex function. For a given y, the subgradient of $f_i(y)$ is given by the Lagrange multiplier λ_i corresponding to the constraint (3f).*

Proof. For simplicity, the notion (t_k) is omitted in this proof. Define the Lagrange function of each agent as

$$L_i(U_i, y, \lambda_i) = J_i(U_i) - \lambda_i^\top \big(y_i(\mathcal{T}) - y\big),$$

where λ_i is the Lagrangian multipliers. Then, consider the dual function

$$d_i(\lambda_i, y) = \min_{U_i \in \bar{\mathscr{U}}_i} L_i(U_i, y, \lambda_i),$$

where $\bar{\mathscr{U}}_i$ is the set of constraints on the control input U_i as $\bar{\mathscr{U}}_i \triangleq \{U_i = \mathrm{col}\{u_i(0), u_i(1), \ldots, u_i(\mathcal{T}-1)\} \in \mathbb{R}^{m_i \mathcal{T}} | \text{satisfying}(3c),(3d)\}$. Recalling (3f) together with Assumption 3, it follows from [30, Proposition 6.2.3] that the strong duality holds, resulting in

$$f_i(y) = \max_{\lambda_i} d_i(\lambda_i, y).$$

Subsequently, $\forall \breve{y} \in \mathscr{Y}, \check{\breve{y}} \in \mathscr{Y}$ and the Lagrangian multiplier $\check{\lambda}_i$ corresponding to $\breve{y}$, there has

$$\begin{aligned}
f_i(\check{\breve{y}}) =& \max_{\lambda_i} \Big\{ \min_{U_i} \big\{ J_i(U_i) - \lambda_i^\top \big(y_i(\mathcal{T}) - \check{\breve{y}}\big)\big\}\Big\} \\
\geq& \min_{U_i} \big\{ J_i(U_i) - \check{\lambda}_i^\top \big(y_i(\mathcal{T}) - \check{\breve{y}}\big)\big\} \\
=& \min_{U_i} \big\{ J_i(U_i) - \check{\lambda}_i^\top \big(y_i(\mathcal{T}) - \breve{y}\big)\big\} + \check{\lambda}_i^\top (\check{\breve{y}} - \breve{y}) \\
=& f_i(\breve{y}) + \check{\lambda}_i^\top (\check{\breve{y}} - \breve{y}).
\end{aligned}$$

It follows from [30, Section 4.2] that $\check{\lambda}_i$ satisfies the definition of the subgradient of $f_i(y)$ at $\breve{y}$. Finally, defining $h_i(\lambda_i) = \min_{U_i} \big\{ J_i(U_i) - \lambda_i^\top y_i(\mathcal{T}) \big\}$ and noticing $f_i(\check{\breve{y}}) = \max_{\lambda_i} \big\{ h_i(\lambda_i) + \lambda_i^\top \check{\breve{y}} \big\}$, then, for any given feasible states $\check{\breve{y}}_1$ and $\check{\breve{y}}_2$ and $0 \leq \theta \leq 1$, there has

$$\begin{aligned}
f_i(\theta\check{\breve{y}}_1 + (1-\theta)\check{\breve{y}}_2) =& \max_{\lambda_i} \big\{ h_i(\lambda_i) + \lambda_i^\top [\theta\check{\breve{y}}_1 + (1-\theta)\check{\breve{y}}_2] \big\} \\
=& \max_{\lambda_i} \Big\{ \theta\big(h_i(\lambda_i) + \lambda_i^\top \check{\breve{y}}_1\big) + (1-\theta)\Big(h_i(\lambda_i) + \lambda_i^\top \check{\breve{y}}_2\big) \Big\} \\
\leq& \theta \max_{\lambda_i} \big\{ h_i(\lambda_i) + \lambda_i^\top \check{\breve{y}}_1 \big\} + (1-\theta) \max_{\lambda_i} \big\{ h_i(\lambda_i) + \lambda_i^\top \check{\breve{y}}_2 \big\}, \\
=& \theta f_i(\check{\breve{y}}_1) + (1-\theta)\check{\breve{y}}_2.
\end{aligned}$$

Therefore, it can be concluded that $f_i(y)$ is convex with respective to y, which completes the proof. □

Proposition 2 *There exists a scalar $\varrho_i > 0$ such that*

$$\|g_i(y)\| \leq \varrho_i, \ \forall g_i(y) \in \partial f_i(y), \ \forall y \in \mathscr{Y}.$$

Proof. Since both $\mathcal{U}_i$ and $\mathcal{X}_i$ are convex, and $\mathcal{U}_i$ is compact for each agent, it follows from (2) and Definition 1 that $\mathscr{Y}$ is compact and convex. Then, it can be concluded that the optimal solution set is non-empty and the subgradient of f_i is bounded. In other words, there exists a scalar $\varrho_i > 0$ which allows the desired conclusion. □

Remark 2 *The above analysis clarifies that the subgradient $\partial f_i(t_k)$ is the Lagrange multiplier λ_i associated with the constraint (3f), which can be obtained by solving the Karush-Kuhn-Tucker conditions [30, Proposition 6.2.5] condition for a given $y(t_k)$ in practice.*

After obtaining the subgradient information $\partial f_i(y(t_k))$, the subgradient-based distributed algorithm is designed in Algorithm 1. Then, (4) can be addressed at each time t_k to obtain $y^\star(t_k)$. The detailed analysis of the convergent properties for the proposed algorithm is provided in Theorem 1.

Remark 3 *The proposed Algorithm 1 is inspired by the dual average method [31] combined with the push-sum technology in [28]. The convergence analysis of Algorithm 1 in Theorem 1 will play an important role to overcome these drawbacks and ensure the stability of the proposed MPC-based method, and the detailed discussion will be provided in Section 4. Compared with other existing methods, the convergence properties of these algorithms in [26,27,29] cannot work for this scenario, which fail to devise termination conditions to give theoretical guarantees.*

Remark 4 *It is possible to use the algorithm in [28] to solve the considered optimization problem (4) within the prediction horizon $\mathcal{T}$, however, it has some drawbacks compared with the proposed Algorithm 1. Specifically, in [28], the subgradient oracle ∂f_i is driven by the sequence $\tilde{y}_i^q$ in step (5b), while its convergence cannot be guaranteed. Although the improvement is discussed in [28], where more additional variables need to be introduced, which increases the resource burden. In contrast, in this paper, the subgradient λ_i from $f_i(y)$ is used directly in (5b) with a rigorous convergence analysis. Additionally, Theorem 1 shows that the constant factor in the convergence rate of Algorithm 1 has been improved compared to that in [28].*

Algorithm 1 Directed Optimal Output Consensus Algorithm within MPC Framework

Initialize initial value $\xi_i^0 \in \mathbb{R}^r, \eta_i^0 = 1, y_i^0 \in \mathscr{Y}, \tilde{y}_i^0 \in \mathscr{Y}$, $\frac{\xi_j^0}{\nu_j}$, $\frac{\eta_j^0}{\nu_j}$, $j \in \mathcal{N}_i^{\text{in}}$, $\gamma > 0$, and $q = 0$ for the ith agent.

1: **repeat**

$$\alpha_q = \frac{\gamma}{\sqrt{q+1}}, \tag{5a}$$

$$\xi_i^{q+1} = \sum_{j \in \mathcal{N}_i^{\text{in}}} \frac{\xi_j^q}{\nu_j} + \lambda_i^q, \tag{5b}$$

$$\eta_i^{q+1} = \sum_{j \in \mathcal{N}_i^{\text{in}}} \frac{\eta_j^q}{\nu_j}, \tag{5c}$$

$$\tilde{y}_i^{q+1} = \arg\min_{\tilde{y}_i \in \mathscr{Y}} \left\{ \langle \frac{\xi_i^{q+1}}{\eta_i^{q+1}}, \tilde{y}_i \rangle + \frac{1}{\alpha_q} \|\tilde{y}_i\|^2 \right\}, \tag{5d}$$

$$y_i^{q+1} = \frac{q}{q+1} y_i^q + \frac{1}{q+1} \tilde{y}_i^q. \tag{5e}$$

2: **communication with neighbors** $\frac{\xi_i^q}{\nu_i}$ and $\frac{\eta_i^q}{\nu_i}$;

3: **set** $q = q + 1$;

4: **until satisfying the predefined termination condition** $\Upsilon_i(y_i^q, y_i^{q+1}) \leq 0$;

5: **return** the optimal consensus output $y_i^\star(t_k) = y_i^\star$.

6: **compute** the optimal control input $U_i^\star(t_k)$ by (3).

Theorem 1 *Concerning the optimization problem (4) at time t_k, the solution sequences $\{y_i^q, q = 0, 1, \ldots\}$ generated by (5) of Algorithm 1 satisfy that*

$$0 \leq F(y_i^q) - F(y^\star) \leq \frac{N}{\sqrt{q}} [(2m_1 + 4M)\varrho\gamma + \varrho^2\gamma + \frac{2\|y^\star\|^2}{\gamma}] + (\frac{3}{q} - \frac{2}{q\sqrt{q}}) N\varrho\gamma(m_2 + m_3), \tag{6}$$

$$\|y_i^q - y_j^q\| \leq \frac{4\gamma M}{\sqrt{q}}, \quad \forall i, j \in \mathcal{V}, \tag{7}$$

where $\varrho, M, m_1, m_2, m_3$ are positive constants, as specified in the proof.

Proof. The proof is given in Appendix A. □

Algorithm 1 shows that the optimal consensus output problem within finite horizon $\mathcal{T}$ can be solve in a distributed manner. It should be noted that in Step 4 of Algorithm 1, the termination condition has been given as a general form $\Upsilon_i(y_i^q, y_i^{q+1})$, which plays an important role in practical applications. More discussions will be provided in the next subsection under the MPC scheme to analyze the steady performance of agents.

3.2 Distributed Optimization with Premature Termination

Within the MPC framework, Problem 1 can be solved and the first portion of the optimal solution $u_i^\star(\tau|t_k)$, $\tau \in [t_k, t_{k+1})$ can be implemented as the real-time input. Theorem 1 shows that the iterative Algorithm 1 ensures a consistent output when the number of iterations q tends to infinity, i.e., $\lim_{q\to\infty} (y_i^q - y_j^q) = \mathbf{0}$, $\forall i, j \in \{1, 2, \ldots, N\}$. However, in practice, there is usually a high demand for real-time performance of implementation with limited resources, which may restrict agents to perform only finite iterations. Therefore, the designed iterative algorithm requires an **appropriate termination condition** to maximize the consensus accuracy and guarantee the closed-loop stability of systems with limited computational power. To illustrate the pattern clearly, the following Figure 1 shows the execution of the MPC-based algorithm.

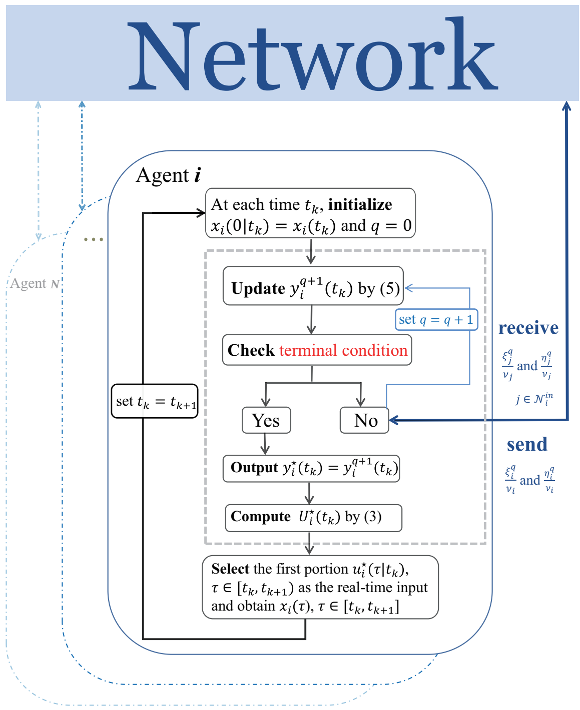

Figure 1. The execution of the MPC-based algorithm

Then, the MPC-based algorithm with premature termination is shown in the following Algorithm 2.

Then, the closed-loop stability of the multi-agent system will be discussed in the next section to demonstrate the effectiveness of the improved MPC-based algorithm.

Remark 5 *The implementation of the aforementioned MPC-based approach is shown in Figure 1 and Algorithm 2, where (8a) and (8b) are the specific termination conditions in Step 4 of Algorithm 1 to make it terminate prematurely.*

Algorithm 2 MPC-based Optimal Output Consensus Algorithm with Premature Termination

Initialize initial value $x_i(0|t_k) = x_i(t_k)$, $\xi_i^0 \in \mathbb{R}^r$, $\eta_i^0 = 1$, $y_i^0, \tilde{y}_i^0 \in \mathscr{Y}$, $f_i(y_i^\star(t_{-1})) = f_i(y_i^0)$, $\frac{\xi_j^0}{\nu_j}$, $\frac{\eta_j^0}{\nu_j}$, $\gamma > 0$, and $q = 0$ for the ith agent at each update time $t_k = 0, \Delta t, 2\Delta t \dots$

1: **repeat**
2: **implement** (5)
3: **communication with neighbors** $\frac{\xi_i^q}{\nu_i}$, $\frac{\eta_i^q}{\nu_i}$ and $\diamond f_i^q(t_k)$, where $\diamond f_i^q(t_k) = f_i(y_i^q(t_k)) - f_i(y_i^\star(t_{k-1}))$;
4: **set** $q = q + 1$;
5: **until** satisfy the terminal conditions

$$\omega_i \diamond f_i^q(t_k) + \sum_{j \in \mathcal{N}_i^{\text{in}}} \frac{1}{\mu_i} \diamond f_j^q(t_k) \leq 0, \tag{8a}$$

$$\frac{4\gamma M}{\sqrt{q}} \leq \varepsilon(t_k), \tag{8b}$$

where $\varepsilon(t_k)$ is a monotonically decreasing function satisfying $\lim_{t_k \to \infty} \varepsilon(t_k) = 0$, and $\omega_i \geq 2$;
6: **set** $y_i^\star(t_k) = y_i^{\tilde{q}}(t_k)$ and **compute** $U_i^\star(t_k)$;
7: **select** the first portion $u_i^\star(\tau|t_k)$, $\tau \in [t_k, t_{k+1})$ as the real-time input.
8: **set** $t_k = t_{k+1}$ and **return** to step 1;

4 Stability Analysis

Before proceeding to the discussion on the steady performance of the multi-agent system, the suboptimal MPC scheme [32] will be firstly introduced, which is essentially the basis of Algorithm 2. Furthermore, the corresponding intuition for the terminal conditions (8a)-(8b) will be explained.

Consider the case that Algorithm 1 stops at the finite $\tilde{q}$th iteration at t_k, resulting in the different $y_i^{\tilde{q}}, \tilde{u}_i^e, \tilde{z}_i^e$, the control input $u_i^{\tilde{q}}(\tau|t_k)$, $\tau \in \{0, 1, \dots, \mathcal{T} - 1\}$ and the system state $x_i^{\tilde{q}}(\tau|t_k)$, $\tau \in \{0, 1, \dots, \mathcal{T}\}$ of each agent. Then, set the candidate sequences at the next time t_{k+1} as $\bar{y}_i(t_{k+1}) = y_i^{\tilde{q}}(t_k)$,

$$\bar{u}_i(\tau|t_{k+1}) = \begin{cases} u_i^{\tilde{q}}(\tau|t_k), & \text{if } \tau \in [t_{k+1}, t_k + \mathcal{T}) \\ \tilde{u}_i^e(t_k), & \text{if } \tau \in [t_k + \mathcal{T}, t_{k+1} + \mathcal{T}) \end{cases}$$

$$\bar{x}_i(\tau|t_{k+1}) = \begin{cases} x_i^{\tilde{q}}(\tau|t_k), & \text{if } \tau \in [t_{k+1}, t_k + \mathcal{T}] \\ \tilde{z}_i^e(t_k), & \text{if } \tau \in [t_k + \mathcal{T} + 1, t_{k+1} + \mathcal{T}] \end{cases}$$

which can be verified that they are feasible solutions of (3) at t_{k+1}. Then, there exists

$$\begin{aligned}
& J(\bar{\mathbf{U}}(t_{k+1})) - J(\mathbf{U}^{\tilde{q}}(t_k)) \\
=& \sum_{i=1}^{N} \left(J_i(\bar{\mathbf{U}}_i(t_{k+1})) - J_i(\mathbf{U}_i^{\tilde{q}}(t_k)) \right) \\
=& \sum_{i=1}^{N} \Bigg\{ \sum_{\tau=t_{k+1}}^{t_{k+1}+\mathcal{T}-1} \left[\|\bar{e}_{xi}(\tau|t_{k+1})\|_{Q_i}^2 + \|\bar{e}_{ui}(\tau|t_{k+1})\|_{R_i}^2 \right] \\
& - \sum_{\tau=t_k}^{t_k+\mathcal{T}-1} \left[\|e_{xi}^{\tilde{q}}(\tau|t_k)\|_{Q_i}^2 + \|e_{ui}^{\tilde{q}}(\tau|t_k)\|_{R_i}^2 \right] \Bigg\} \\
\leq& - \sum_{i=1}^{N} \Bigg\{ \sum_{\tau=t_k}^{t_{k+1}-1} \left[\|e_{xi}^{\tilde{q}}(\tau|t_k)\|_{Q_i}^2 + \|e_{ui}^{\tilde{q}}(\tau|t_k)\|_{R_i}^2 \right] \Bigg\} \\
\leq& 0,
\end{aligned} \tag{9}$$

where $\bar{e}_{xi}(\tau|t_{k+1}) = \bar{x}_i(\tau|t_{k+1}) - \tilde{z}_i^e(t_{k+1})$, $\bar{e}_{ui}(\tau|t_{k+1}) = \bar{u}_i(\tau|t_{k+1}) - \tilde{u}_i^e(t_{k+1})$, $e_{xi}^{\tilde{q}}(\tau|t_k) = x_i^{\tilde{q}}(\tau|t_k) - \tilde{z}_i^e(t_k)$, and $e_{ui}^{\tilde{q}}(\tau|t_k) = u_i^{\tilde{q}}(t_k) - \tilde{u}_i^e(\tau|t_k)$. Referring to the suboptimal MPC scheme [32], the above inequality (9) shows the improved property of the objective function, which is significant to guarantee stability. Thus, a termination condition is required to be established for testing the improvement. Then, considering the communication between the agents, a distributed condition (8a) is designed in Algorithm 2 to determine whether the improved property (9) is satisfied or not. And the detailed verification is summarized in the proof of Theorem 2.

However, due to the finite iteration $\tilde{q}$ of Algorithm 1, it will cause agents to converge to different equilibrium points, thus failing to achieve consensus output, i.e., $y_i^{\tilde{q}} \neq y_j^{\tilde{q}}$ for some $i \neq j$. To overcome this dilemma, it is useful to recall the property of Algorithm 1. It follows from (7) that the difference between the consensus output y_i will decrease as the iteration increases. Intuitively, it is possible to consider the terminal condition (8b) consisting of an upper bound $\varepsilon(t_k)$ to restrict the difference between the output of the agents, which satisfies $\|y_i^q(t_k) - y_j^q(t_k)\| \leq \varepsilon(t_k) \to 0$ as $t_k \to \infty$. Then, the asymptotic convergence of the consensus output can be guaranteed.

The above observations clarify the inspiration and importance of terminal conditions (8a)-(8b). Then, the closed-loop stable analysis of Algorithm 2 is provided in the following theorem.

Theorem 2 *The output consensus of the multi-agent system (1) can be achieved asymptotically by using Algorithm 2.*

Proof. First, define

$$\diamond \mathbf{f}^q(t_k) = \text{col}\{\diamond f_1^q(t_k), \diamond f_2^q(t_k), \dots, \diamond f_N^q(t_k)\},$$
$$V_i^q(t_k) = \omega_i \diamond f_i^q(t_k) + \sum_{j \in \mathcal{N}_i^{\text{in}}} \frac{1}{\mu_i} \diamond f_j^q(t_k), \ i \in \mathcal{V},$$

and $\mathbf{V}^q(t_k) = \text{col}\{V_1^q(t_k), V_2^q(t_k), \dots, V_N^q(t_k)\}$. Clearly, $\mathbf{V}^q(t_k)$ can be written as $\mathbf{V}^q(t_k) = \mathcal{D} \diamond \mathbf{f}^q(t_k)$, where $\mathcal{D}$ can be proved to be a diagonal dominate matrix owing to $\omega_i \geq 2$. Therefore, $\mathcal{D}$ is an invertible matrix and all elements of $\mathcal{D}^{-1}$ are non-negative due to the Cramer's rule [33, chapter 0.8.3]. Then, it follows from (8a) that

$$\sum_{i=1}^{N} \diamond f_i^q(t_k) = \mathbf{1}^\top \diamond \mathbf{f}^q(t_k) = \mathbf{1}^\top \mathcal{D}^{-1}\mathbf{V}^q(t_k) \leq 0, \quad (10)$$

which indicates the improved performance (9) and ensures each individual to reach its y_i asymptotically. Recalling (7), then, it can be concluded that (8b) ensures $\lim_{t_k\to\infty}\left(y_i^q(t_k) - y_j^q(t_k)\right) = \mathbf{0}$, $\forall i, j \in \mathcal{V}$, thereby enabling the multi-agent system to achieve consensus asymptotically. □

Remark 6 *There are two termination conditions that need to be checked in (8). As described above, (8a) indicates the improvement of the system performance, and (8b) ensures that the convergence of the consensus output. For the upper bound $\varepsilon(t_k)$ in (8b), for example, it can be set as $\varepsilon(t_k) = M\sqrt{\frac{\varepsilon_0}{t_k+1}}$ with constant $\varepsilon_0 > 0$. Then (8b) can be replaced by $q \geq \frac{16\gamma^2(t_k+1)}{\varepsilon_0}$ owing to (7). Moreover, if the communication graph is a complete graph, then there exists at least one node that has access to $\diamond f_i^q(t_k)$ of all agents, and the performance condition (8a) can be replaced by $\sum_{i=1}^{N} \diamond f_i^q(t_k) \leq 0$ to be easily checked.*

Remark 7 *In practice, there is a trade-off in the number of iterations q at each prediction time. In general, the larger q is, the greater the accuracy of the solution obtained by Algorithm 1, which can enable the agents to take less time t_k to achieve consensus. However, considering the limitation on computational resources, (8) gives a lower bound on the number of iterations that guarantees the asymptotic consensus output. Besides, when the agents' outputs are close (which may happen when the time t_k is large), the decentralized MPC scheme can be established with fixed y_i, $i \in \{1, 2, \ldots, N\}$ to further conserve resources.*

Remark 8 *If the restriction of the finite iteration number is not considered in the ideal case, that is, Algorithm 1 can execute the infinite iterations to converge to the optimal solution $y_i^\star(t_k), U_i^\star(t_k)$, then the termination condition (8) of Algorithm 2 can be neglected. In this way, Algorithm 2 will degenerate into the standard MPC method and the closed-loop stability of the system can be similarly ensured by the previous analysis, which is omitted here for brevity.*

5 Simulation Example

In this section, the effectiveness and superiority of the proposed algorithm will be verified by testing it on different scenarios.

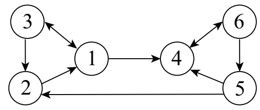

Figure 2. Communication topology of agents

Part I. First, the proposed algorithm is applied to solve the optimal formation problem for six agents under a directed topology as shown in Figure 2.
Suppose the system contains three agents $i \in \{1, 2, 3\}$ with first-order dynamics

$$p_i(k+1) = p_i(k) + \delta_i u_i(k) \quad (11)$$

and three agents with second-order dynamics

$$\begin{aligned} p_i(k+1) &= p_i(k) + \delta_i v_i(k), \\ v_i(k+1) &= v_i(k) + \delta_i u_i(k) \end{aligned} \quad (12)$$

for $i \in \{4, 5, 6\}$, where $p_i = [p_{xi}, p_{yi}]^\top \in \mathbb{R}^2$, $v_i = [v_{xi}, v_{yi}]^\top \in \mathbb{R}^2$, $u_i = [u_{xi}, u_{yi}]^\top \in \mathbb{R}^2$ and δ_i is the sample period, $i \in \{1, 2, \ldots, 6\}$.

Set p_i as the output y_i, which can be regard as the physical position of each individual. The goal is to complete the optimal formation task, and the constraint (3f) is modified as $y_i(\mathcal{T}|t_k) = y(t_k) + d_{ci}$, where d_{ci} denotes the relative position between agent i and the consistent output in the desired formation pattern. Assume the formation pattern is defined as $[d_{c1}, d_{c2}, \ldots, d_{c6}] = [(-1, \sqrt{3})^\top, (-2, 0)^\top, (-1, -\sqrt{3})^\top, (1, \sqrt{3})^\top, (2, 0)^\top, (1, -\sqrt{3})^\top]$, which forms a regular hexagon. Then, assume the constraints on system outputs and control inputs are $p_i \in \mathcal{P}_i \triangleq \{p_i | [-6, -6]^\top \leq p_i \leq [6, 6]^\top\}$ $u_i \in \mathcal{U}_i \triangleq \{u_i | [-3, -3]^\top \leq u_i \leq [3, 3]^\top\}$, respectively. The weighting matrices are set as $Q_i = I_2$ and $R_i = 0.1 I_2$, and other auxiliary parameters are set as $\gamma = 0.1$, $\varepsilon(t_k) = \frac{M}{150\sqrt{t_k+1}}$, $\xi_i^0 = \mathbf{0}$, and $y_i^0 = \tilde{y}_i^0 = \mathbf{0}$. The initial state of each individual is set as $p_{xi}(0) = \text{rand}(-6, 6)$, $p_{yi}(0) = \text{rand}(-6, 6)$, and the sample period is defined as $\delta_i = \text{rand}(0.8, 1.0)$s, while the prediction horizon is $\mathcal{T} = 8$ and the control period is $\Delta t = 1$.

The output of and the control input of the multi-agent system are recorded as shown in Figure 3. It can be seen that agents reach the desired output state rapidly, while the control inputs are damped quickly, both of which satisfy the saturation constraints. The spatio-temporal trajectories of agents generated by the proposed algorithm are shown in Figure 4, which illustrate the efficacy of the MPC-based method.

To show the effectiveness of the premature termination (8), the implementation of Algorithm 2 without this premature termination is simulated for comparison, where the stopping condition is set as $\|y_i^q - y_i^{q+1}\| \leq e_i$. Here, two cases of $e_i = 10^{-1}$ and $e_i = 10^{-5}$ are considered.

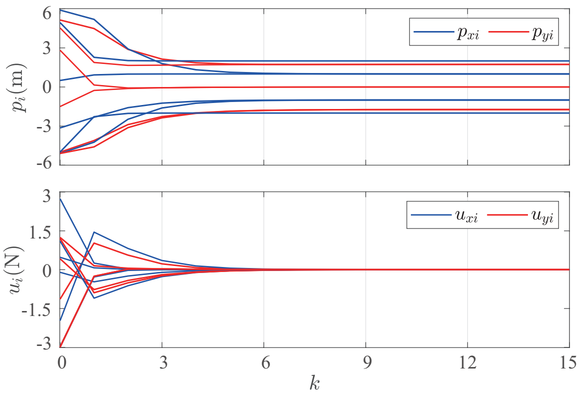

Figure 3. Output and control input of each agent

When $e_i = 10^{-1}$, agents fail to accomplish the required formation due to the large error caused by the aggressive stopping condition. On the contrary, when $e_i = 10^{-5}$, both of this case and the proposed Algorithm 2 accomplish the formation task. However, the total number of iterations of this case is $q_{sum} = 9097$, which is much larger than the proposed Algorithm 2 with $q_{sum} = 1089$ since the stopping condition is conservative. The comparison result shows that the proposed Algorithm 2 with premature termination can effectively reduce the number of iterations without loss of control performance, thereby demonstrating its effectiveness.

Part II. To test the algorithm comprehensively, a batch of random networked systems are simulated for the optimal consensus problem. Consider the cases where the total number of agents are $N = 60$ and $N = 120$, and randomly generate 100 directed connected graphs for both cases. On this basis, consider the dynamics of the individuals are all (11), or all (12), or a randomized mixture of (11) and (12), where the dimension of the control input $m = 1$ and $m = 2$ are taken into account, respectively. For comparison, other control schemes will be considered to show the superiority of the proposed algorithm. Here, two system transformation-based consensus methods, namely [6] (STC-I) and [7] (STC-II) are involved. Then, define the consensus time instant $\mathscr{T}$ satisfying $\left\| \sum_{i=1}^N \sum_{j=1}^N \mathcal{A}_{ij}\big(p_i(\mathscr{T}) - p_j(\mathscr{T})\big) \right\| < 10^{-3}$, and the performance cost function as $\bar{J} = \sum_{i=1}^N \sum_{t=0}^{\mathscr{T}-1} \left[\sum_{j=1}^N \mathcal{A}_{ij} \|p_i(t) - p_j(t)\|_{Q_i}^2 + \|u_i(t)\|_{R_i}^2 \right]$.

The comparison results are shown in Figure 5, where the average time and average cost of each case are obtained by averaging the consensus time $\mathcal{T}$ and the performance cost $\bar{J}$ of 100 random networks, respectively. Simulation results demonstrate that the proposed algorithm has better consensus performance.

6 Conclusion

In this paper, the optimal consensus output problem of multi-agent systems over directed graphs is investigated

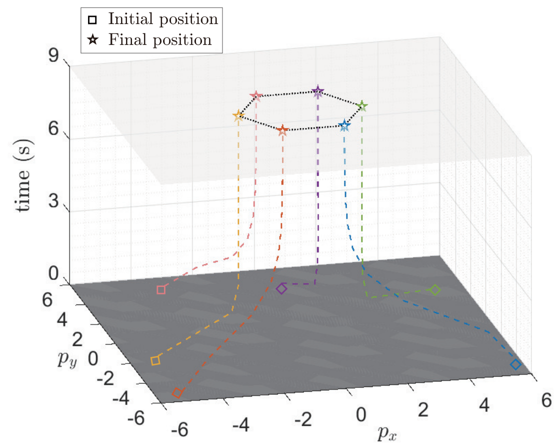

Figure 4. Spatio-temporal trajectories of agents.

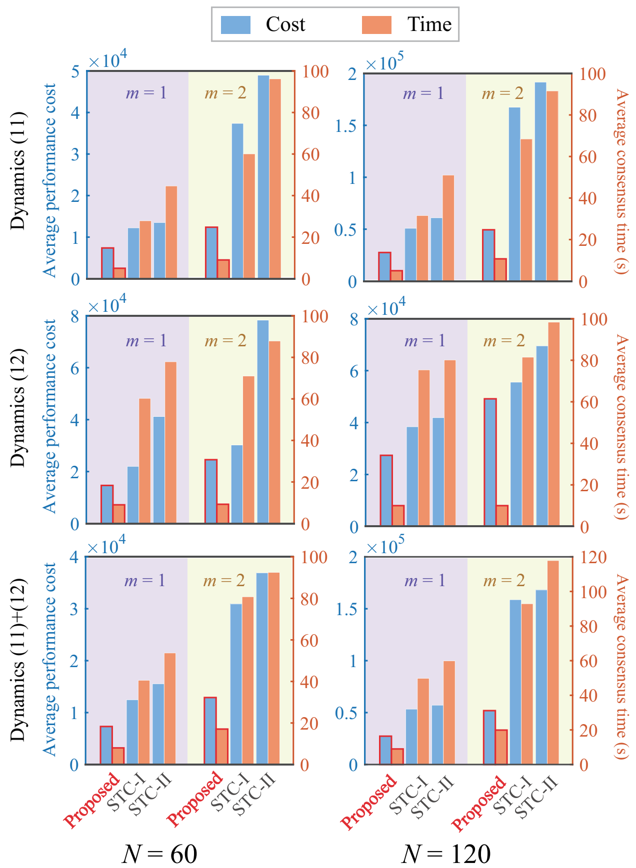

Figure 5. Comparison between different methods.

within the DMPC framework. Drawing on the primal decomposition technique and the push-sum dual average method, an iterative algorithm for directed graphs is first given to solve the optimization problem at each time. Furthermore, a finite iteration MPC-based approach is established, enhancing the effectiveness in practical applications. The stability of the improved algorithm is analyzed using suboptimal MPC theory, deriving terminal conditions to guarantee asymptotic consensus of the multi-agent system. The theoretical results and the superiority of the proposed algorithm are finally verified by numerical simulation examples.

Acknowledgements

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A Proof of Proposition 2

Before proceeding to the proof, some useful definitions and lemmas are prepared.

Definition 3 For $\lambda_i^q \in \partial f_i(y)$ and $\tilde{\lambda}_i^q \in \partial f_i(\tilde{y})$, define $\beta^q = \frac{1}{N}\sum_{i=1}^N \lambda_i^q$, $\tilde{\beta}^q = \frac{1}{N}\sum_{i=1}^N \tilde{\lambda}_i^q$,
$\psi^{q+1} = \arg\min_{y\in\mathscr{Y}} \left\{\langle \sum_{s=0}^q \beta^s, \alpha_q\rangle + \frac{1}{\alpha_q}\|y\|^2\right\}$,
$\tilde{\psi}^{q+1} = \arg\min_{y\in\mathscr{Y}} \left\{\langle \sum_{s=0}^q \tilde{\beta}^s, \alpha_q\rangle + \frac{1}{\alpha_q}\|y\|^2\right\}$.

Lemma 1 [28, Lemma 4.3] For any $i \in \{1, 2, \ldots, N\}$, there holds $\|\tilde{y}_i^q - \psi^q\| \le \alpha_{q-1}\left\|\frac{\xi_i^{q+1}}{\eta_i^{q+1}} - \sum_{s=0}^q \beta^s\right\|$.

Lemma 2 [28, Lemma 4.2] Consider the weighted adjacency matrix $\tilde{\mathcal{A}}$ defined as $\left[\tilde{\mathcal{A}}\right]_{ij} = \frac{1}{d_j}$, if $j \in \mathcal{N}_i^{\text{in}}$, and $\left[\tilde{\mathcal{A}}\right]_{ij} = 0$, otherwise. Define $\mathcal{M}_1^q = \sum_{j=1}^N \left[\tilde{\mathcal{A}}^q\right]_{ij} \xi_j^0/\eta_i^{q+1}$, $\mathcal{M}_2^q = \sum_{s=0}^q \sum_{j=1}^N \left(\left[\tilde{\mathcal{A}}^{q-s-1}\right]_{ij} - \phi_i^q\right)\lambda_j^s/\eta_i^{q+1}$, $\mathcal{M}_3^q = \sum_{s=0}^q \sum_{j=1}^N \left(\phi_i^q \lambda_j^s - \frac{\eta_i^{q+1}}{N}\right)\lambda_j^s/\eta_i^{q+1}$, which satisfy $\|\mathcal{M}_1^q\| \le m_1$, $\|\mathcal{M}_2^q\| \le m_2$, $\|\mathcal{M}_3^q\| \le m_3$ with the positive scalars m_1, m_2, m_3. Then, for $M = m_1 + m_2 + m_3$, there has $\|\frac{\xi_i^{q+1}}{\eta_i^{q+1}} - \sum_{s=0}^q \beta^s\| = \|\mathcal{M}_1^q + \mathcal{M}_2^q + \mathcal{M}_3^q\| \le M$.

Lemma 3 Considering $\alpha_s = \frac{\gamma}{\sqrt{s+1}}$, there has
(a) $\frac{1}{q}\sum_{s=1}^q \alpha_{s-1} = \frac{1}{q}\sum_{s=1}^q \frac{\gamma}{\sqrt{s}} \le \frac{2\gamma}{\sqrt{q}}$,
(b) $\frac{1}{q}\alpha_q = \frac{\sqrt{q+1}}{\gamma q} \le \frac{2}{\gamma\sqrt{q}}$,
(c) $\frac{1}{q}\sum_{s=1}^q \frac{\alpha_{s-1}}{s} = \frac{1}{q}\sum_{s=1}^q \frac{\gamma}{s\sqrt{s}} \le \frac{\gamma}{q}(3 - \frac{2}{\sqrt{q}})$.

Proof. The conclusion (a) holds due to the fact that $\frac{1}{q}\sum_{s=1}^q \frac{\gamma}{\sqrt{s}} \le \frac{1}{q}\sum_{s=1}^q \frac{2\gamma}{\sqrt{s}+\sqrt{s-1}}$, while (b) holds owing to $q > 0$. As for (c), there holds that $\frac{1}{q}\sum_{s=1}^q \frac{\gamma}{s\sqrt{s}} \le \frac{\gamma}{q}(1 + \int_1^q \frac{1}{s\sqrt{s}}ds) = \frac{\gamma}{q}(3 - \frac{2}{\sqrt{q}})$. □

Lemma 4 [31, Lemma 2] For the optimal solution $y^\star$, there has $\sum_{s=1}^q \langle\tilde{\beta}^s, \tilde{\psi}^s - y^\star\rangle \le \frac{1}{2}\sum_{s=1}^q \alpha_{s-1}\|\tilde{\beta}^s\|^2 + \frac{1}{\alpha_q}\|y^\star\|^2$.

Then, the proof of Proposition 2 is presented as follows.

First, recalling the update rule (5e), there has $y_i^q = \frac{1}{q}\sum_{s=1}^q \tilde{y}_i^q$, and one can obtain

$$0 \le F(y_i^q) - F(y^\star) \le \frac{1}{q}\sum_{s=1}^q \left(F(\tilde{y}_i^s) - F(y^\star)\right) \quad \text{(A.1)}$$

$$\|y_i^q - y_j^q\| \le \frac{1}{q}\sum_{s=1}^q \|\tilde{y}_i^s - \tilde{y}_j^s\|. \quad \text{(A.2)}$$

By Lemma 1 and Lemma 2, there has

$$\|\tilde{y}_i^q - \psi^q\| \le \alpha_{q-1}\left\|\frac{\xi_i^{q+1}}{\eta_i^{q+1}} - \sum_{s=0}^q \beta^s\right\| \le M\alpha_{q-1}, \quad \text{(A.3)}$$

and $\|\tilde{y}_i^q - \tilde{y}_j^q\| \le \|\tilde{y}_i^q - \psi^q\| + \|\tilde{y}_j^q - \psi^q\| \le 2M\alpha_{q-1}$. Define $\tilde{F}_1 = \frac{1}{q}\sum_{s=1}^q \left(F(\tilde{y}_i^s) - F(\tilde{y}_j^s)\right)$, which satisfies

$$\tilde{F}_1 \le \frac{1}{q}\sum_{s=1}^q N\rho\|\tilde{y}_i^s - \tilde{y}_j^s\| \le 2\alpha_{q-1}MN\rho\frac{1}{q}\sum_{s=1}^q 2M\alpha_{s-1} \le 4\rho MN\gamma/\sqrt{q}. \quad \text{(A.4)}$$

Furthermore, it follows from (A.2) that

$$\|y_i^q - y_j^q\| \le \frac{1}{q}\sum_{s=1}^q \|\tilde{y}_i^s - \tilde{y}_j^s\| \le \frac{1}{q}\sum_{s=1}^q 2M\alpha_{s-1}.$$

Then, utilizing Lemma 3 (a)-(b), the desired conclusion (7) can be proved.

Recalling Definition 3 and according to the Chain Rule [30, Proposition 4.2.5], there has $\tilde{\beta}^q = \frac{1}{q}\beta^q$. Then, similar to the process in Lemma 2 and (A.3), there holds

$$\|\tilde{y}_i^q - \tilde{\psi}^q\| \le \alpha_{q-1}(m_1 + \frac{m_2}{q} + \frac{m_3}{q}).$$

Defining $\tilde{F}_2^A = \frac{1}{q}\sum_{s=1}^q \sum_{j=1}^N \langle\tilde{\lambda}_j^s, \tilde{y}_j^s - \tilde{\psi}^s\rangle$, there has

$$\tilde{F}_2^A \le \frac{1}{q}\sum_{s=1}^q \sum_{j=1}^N \|\tilde{\lambda}_j^s\|\|\tilde{y}_j^s - \tilde{\psi}^s\| \le \rho N\frac{1}{q}\sum_{s=1}^q \alpha_{s-1}(m_1 + \frac{m_2}{s} + \frac{m_3}{s}). \quad \text{(A.5)}$$

Moreover, defining $\tilde{F}_2^B = \frac{1}{q}\sum_{s=1}^q \sum_{j=1}^N \langle\tilde{\lambda}_j^s, \tilde{\psi}^s - y^\star\rangle$ and recalling Lemma 4, there holds

$$\tilde{F}_2^B \le N\frac{1}{q}\left(\frac{1}{2}\sum_{s=1}^q \alpha_{s-1}\|\tilde{\beta}^s\|^2 + \frac{1}{\alpha_q}\|y^\star\|^2\right). \quad \text{(A.6)}$$

Then, for $\tilde{F}_2 = \frac{1}{q}\sum_{s=1}^q \left(F(\tilde{y}_j^s) - F(y^\star)\right)$, there follows

$$\tilde{F}_2 \le \tilde{F}_2^A + \tilde{F}_2^B,$$

$$\leq \frac{1}{q} 3\gamma\rho N(m_2+m_3) - \frac{2}{q\sqrt{q}}\gamma\rho N(m_2+m_3) + \frac{1}{\sqrt{q}}\left(2\gamma\rho N m_1 + \gamma N\rho^2 + 2N\frac{\|y^\star\|^2}{\gamma}\right) \quad (A.7)$$

by using Lemma 3 (c). Finally, combing (A.1), (A.4) and (A.7), the desired conclusion (6) has been proved. □

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